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Traveling along horizontal broken geodesics of a homogeneous Finsler submersion ☆

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ABSTRACT

In this paper, we discuss how to travel along horizontal broken geodesics of a homogeneous Finsler submersion, i.e., we study, what in Riemannian geometry was called by Wilking, the dual leaves. More precisely, we investigate the attainable sets $\mathcal{A}_q(\mathcal{C})$ of the set of analytic vector fields $\mathcal{C}$ determined by the family of horizontal unit geodesic vector fields $\mathcal{C}$ to the fibers $\mathcal{F} = \{\rho^{-1}(c)\}$ of a homogeneous analytic Finsler submersion $\rho : M \to B$. Since reverse of geodesics don't need to be geodesics in Finsler geometry, one can have examples on non compact Finsler manifolds M where the attainable sets (the dual leaves) are no longer orbits or even submanifolds. Nevertheless we prove that, when M is compact and the orbits of $\mathcal{C}$ are embedded, then the attainable sets coincide with the orbits. Furthermore, if the flag curvature is positive then M coincides with the attainable set of each point. In other words, fixed two points of M, one can travel from one point to the other along horizontal broken geodesics.

In addition, we show that each orbit $\mathcal{O}(q)$ of $\mathcal{C}$ associated to a singular Finsler foliation coincides with M, when the flag curvature is positive, i.e., we prove Wilking's result in Finsler context. In particular we review Wilking's transversal Jacobi fields in Finsler case.

1. Introduction

Given a Riemannian submersion $\rho : M \to B$ and the Riemannian foliation $\mathcal{F} = \{\rho^{-1}(c)\}_{c\in B}$, we can associate to it the so called *dual foliation* $\mathcal{F}^{\#} = \{L_q^{\#}\}$, where each leaf of $L_q^{\#} \in \mathcal{F}^{\#}$ is defined as the set of points $x \in M$ that are the end point of a piece-wise smooth horizontal geodesic starting at q.

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In [13] Wilking proved that Sharafutdinov retraction is smooth using the dual foliation of the metric retraction onto the soul. He also proved that if the curvature is positive then M coincides with one single leaf of $\mathcal{F}^{\#}$. In fact this result was proved in the more general context of singular Riemannian foliations.

Dual foliations can also be seen from the point of view of geometric control theory. We can consider the set of smooth vector fields $\mathcal{C} = \{\vec{f}_u\}_{u\in U}$ determined by the family of horizontal unit geodesic vector fields $\vec{f}_u$, i.e., those whose integral curves are horizontal unit speed geodesic segments. With this approach, $L_q^{\#}$ is the attainable set $\mathcal{A}_q(\mathcal{C})$. Since in Riemannian geometry this controls system is symmetric (if $\vec{f}_u \in \mathcal{C}$ then $-\vec{f}_u \in \mathcal{C}$), each orbit $\mathcal{O}(q)$ coincides with $\mathcal{A}_q(\mathcal{C})$, and in particular the leaf $L_q^{\#}$ is an immersed submanifold.

When we consider this discussion in the broader context of Finsler geometry, we see a significant conceptual shift. As illustrated in the simple Example 2.25, $L_q^{\#} = \mathcal{A}_q(\mathcal{C})$ does not need to be a submanifold, since it does not need to coincide with the orbit $\mathcal{O}(q)$. In fact the set of smooth vector fields $\mathcal{C} = \{\vec{f}_u\}$ may no longer be symmetric.

In this paper, we investigate the attainable set $\mathcal{A}_q(\mathcal{C})$ of the set of smooth vector fields $\mathcal{C}$ determined by the family of horizontal unit geodesic vector fields to the fibers $\mathcal{F} = \{\rho^{-1}(c)\}$ of a homogeneous analytic Finsler submersion $\rho : M \to B$.

Theorem 1.1. *Let $\mu : G \times M \to M$ be an analytic Finsler action on an analytic compact Finsler manifold M. Assume that G is compact and that the orbits are principal. Let $\rho : M \to M/G = B$ be the Finsler submersion describing the homogeneous foliation $\mathcal{F} = \{\rho^{-1}(c)\}$ and $\mathcal{C} = \{\vec{f}_u\}_{u\in U}$ be the set of horizontal unit geodesic vector fields associated to the submersion ρ. Then*

(a) If the orbit $\mathcal{O}(q)$ is embedded then it coincides with the attainable set $\mathcal{A}_q(\mathcal{C})$.
(b) If (M, F) has non negative flag curvature and the flag curvature at one point q_0 is $K(q_0) > 0$ then $\mathcal{A}_q(\mathcal{C}) = \mathcal{O}(q) = M$ for each $q \in M$.

With this nice and simple result we hope to stress to an audience of mathematicians with good knowledge on Riemannian geometry, how natural is the relation between geometric control theory and Finsler geometry. In particular, we try to write this note in a self contained presentation. Theorem 1.1 also motivates natural questions that could be explored in future research. The first one is how to generalize it, dropping for example the condition of homogeneity of the submersion (see Remark 3.3) or even considering the most general situation, i.e., dual foliations of singular Finsler foliation; see [2]. A more involving question is whether dual foliations could be used to prove smoothness of class of Finsler submetries, as they were used in Wilking's work in the Riemannian case; see [13].

Item (b) of Theorem 1.1 follows direct from item (a) and from the Proposition 1.2 below, that assures (as in the Riemannian case) that the orbits of the set of horizontal unit geodesic vector fields associated to a singular Finsler foliation (e.g., partition of M into orbits of a Finsler action) coincide with M when flag curvature is positive. Note nevertheless that Proposition 1.2 does not deal with attainable sets, that seems to us the main subject of study in the Finsler case.

Proposition 1.2. *Let (M, F) be a complete Finsler manifold with non negative flag curvature and $\mathcal{F} = \{L\}$ a singular Finsler foliation. Let $\mathcal{C} = \{\vec{f}_u\}$ be the set of horizontal unit geodesic vector fields of $\mathcal{F}$. Assume that there exists a regular leaf L_q so that each point of this leaf has positive flag curvature $K > 0$. Then the orbit $\mathcal{O}(q)$ of $\mathcal{C}$ coincides with M.*

This paper is organized as follows: In Section 2 we review a few facts on geometric control theory and Finsler geometry that will be used in this paper. Item (a) of Theorem 1.1 is proved in Section 3. Proposition 1.2 (and hence item (b) of Theorem 1.1) is proved in Section 4, accepting a few facts on Wilking's transversal Jacobi fields and the Jacobi triples in Finsler case, that are revised in Section 5. In particular, we hope that the review presented in Section 5 allows Finslerian geometers to come into contact with a tool that has been useful in the study of Riemannian submersions and (singular) Riemannian foliations.

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2. Background

2.1. A few facts on geometric control theory

Here we review a few results and definitions on geometric control theory extracted from the classical book of Agrachev and Sachkov [1, Chapters 5, 8] and [11,12].

Let N be a manifold and $\mathcal{C} = \{\vec{f}_u\}$ be a set of smooth (analytic) vector fields *everywhere defined,* i.e., the union of the domains of elements of $\mathcal{C}$ is N. This condition will be always used in this paper.

The *attainable set* of the family $\mathcal{C}$ through q is defined as:

$$\mathcal{A}_q(\mathcal{C}) = \{e^{t_k \vec{f}_k} \circ \cdots \circ e^{t_1 \vec{f}_1}(q),\, t_i \geq 0,\, k \in \mathbb{N},\, \vec{f}_i \in \mathcal{C}\}$$

where $e^{t\vec{f}_i}$ is the flow of $\vec{f}_i \in \mathcal{C}$ in instant t. The *orbit* of the family $\mathcal{C}$ through q is

$$\mathcal{O}(q) = \{e^{t_k \vec{f}_k} \circ \cdots \circ e^{t_1 \vec{f}_1}(q),\, t_i \in \mathbb{R},\, k \in \mathbb{N},\, \vec{f}_i \in \mathcal{C}\}$$

Orbits have nice structures as we see in the next result.

Theorem 2.1 *(Nagano-Stefan-Sussmann). For a given set of vector fields $\mathcal{C}$ everywhere defined on a smooth manifold N, the partition $\{\mathcal{O}(q)\}_{q \in N}$ is a singular foliation, i.e.,*

(a) *each orbit is an immersed submanifold;*
(b) *for each $v_q \in T_q\mathcal{O}(q)$ there exists a vector field $\vec{v}$ on N so that $\vec{v}(q) = v_q$ and $\vec{v}(p) \in T_p\mathcal{O}(p), \forall p \in N$.*

Recall that when the leaves of a singular foliation have the same dimension, the singular foliation is called *regular foliation* or just *foliation.*

Set $\mathrm{Lie}(\mathcal{C}) := \mathrm{Span}\{[\vec{f}_1, [\ldots, [\vec{f}_{k-1}, \vec{f}_k] \ldots]], \vec{f}_i \in \mathcal{C}, k \in \mathbb{N}\}$. With this concept we can establish conditions under which the orbit coincides with the manifold.

Corollary 2.2 *(Rashevsky-Chow). Let N be a connected manifold and $\mathcal{C}$ a set of vector fields. If $\mathrm{Lie}_q(\mathcal{C}) = T_qN, \forall q \in N$, then $N = \mathcal{O}(q), \forall q \in N$.*

A submodule $\mathcal{V}$ (e.g., $\mathcal{V} = \mathrm{Lie}(\mathcal{C})$) is *locally finitely generated over* $C^\infty(N)$, if for each point q, there exists a neighborhood U of q and vector fields $\vec{v}_1, \cdots, \vec{v}_k$ of $\mathcal{V}$ with domain containing U so that $\mathcal{V}|_U = \{\sum_{i=1}^k a_i \vec{v}_i | a_i \in C^\infty(U)\}$.

Remark 2.3. *if a module $\mathcal{V}$ is generated by analytic vector fields, it is locally finitely generated.* This fact makes it possible to use relevant results on attainable sets and it is the main reason why we assume analyticity in Theorem 1.1.

Proposition 2.4. *Let N be a manifold and $q_0 \in N$. If $\mathrm{Lie}(\mathcal{C})$ is locally finitely generated over $C^\infty(N)$, (in particular when $\mathcal{C}$ and N are analytic) then $\mathrm{Lie}_q(\mathcal{C}) = T_q\mathcal{O}(q_0)$ for $q \in \mathcal{O}(q_0)$ and for all orbits $\mathcal{O}(q_0)$.*

Different from orbits, the attainable sets do not need to be immersed submanifolds. But in the case where $\mathrm{Lie}(\mathcal{C})$ is locally finitely generated (e.g., $\mathcal{C}$ is analytic), they still have some interesting properties.

Theorem 2.5 *(Krener). If* $\mathrm{Lie}(\mathcal{C})$ *is locally finitely generated, then* $\mathrm{int}\big(\mathcal{A}_q(\mathcal{C})\big)$ *is dense in* $\mathcal{A}_q(\mathcal{C}) \subset \mathcal{O}(q)$*. Here the density is with respect to the topology of* $\mathcal{O}(q)$*. In particular* $\mathrm{int}\big(\mathcal{A}_q(\mathcal{C})\big) \neq \emptyset$*.*

Let us now move towards results that will allow us to conclude that $\mathcal{A}_{q_0}(\mathcal{C}) = \mathcal{O}(q_0)$ under suitable hypotheses.

Definition 2.6. Given a complete vector field $\vec{g}$ on N, a point $q \in N$ is called *Poisson stable* for $\vec{g}$ if for any $t_0 > 0$ and any neighborhood W of q there exists a point $x \in W$ and a time $t_1 > t_0$ such that $e^{t_1\vec{g}}(x) \in W$. The vector field $\vec{g}$ is *Poisson stable* if all points are Poisson stable.

Proposition 2.7 *(Poincaré). Assume that* N *is compact and the flow* $e^{t\vec{g}}$ *of a complete vector field* $\vec{g}$ *preserves a volume of* N*. Then* $\vec{g}$ *is Poisson stable.*

Definition 2.8. A complete vector field $\vec{f}$ tangent to the orbits of $\mathcal{C}$ is called *compatible with* $\mathcal{C}$ if $\mathcal{A}_q(\mathcal{C})$ is dense in $\mathcal{A}_q(\mathcal{C} \cup \vec{f})$, with respect to the topology of the orbits.

Proposition 2.9. *Assume that* $\mathrm{Lie}(\mathcal{C})$ *is locally finitely generated. If a complete vector field* $\vec{g} \in \mathcal{C}$ *is Poisson stable in the orbit associated to* $\mathcal{C}$*, then* $-\vec{g}$ *is compatible with* $\mathcal{C}$*.*

Proposition 2.10. *Assume that* $\mathrm{Lie}(\mathcal{C})$ *is locally finitely generated. If* $\mathcal{A}_{q_0}(\mathcal{C})$ *is dense in* $\mathcal{O}(q_0)$ *then* $\mathcal{A}_{q_0}(\mathcal{C}) = \mathcal{O}(q_0)$*.*

2.2. A few facts on Finsler geometry

In this section, we briefly review a few facts on Finsler geometry and Finsler submersions necessary to our paper, most of them extracted from [8], [2], [3] and [5]. A comprehensive introduction to this rich geometry can be found in [10].

2.2.1. The metric structure

Definition 2.11. Let M be a manifold. A continuous function $F : TM \to [0, +\infty)$ is called *Finsler metric* if

(a) F is smooth on $TM \setminus \{0\}$,
(b) F is positive homogeneous of degree 1, that is, $F(\lambda v) = \lambda F(v)$ for every $v \in TM$ and $\lambda > 0$,
(c) for every $p \in M$ and $v \in T_pM \setminus \{0\}$, the *fundamental tensor* of F defined as

$$g_v(u, w) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s} F^2(v + tu + sw)|_{t=s=0}$$

for any $u, w \in T_pM$ is a nondegenerate positive-definite bilinear symmetric form, i.e., an inner product.

In particular, if V is a vector space and $F : V \to \mathbb{R}$ is a function smooth on $V \setminus \{0\}$ and satisfying the properties (b) and (c) above, then (V, F) is called a *Minkowski space.*

The fundamental tensor satisfies a few relevant properties.

Proposition 2.12. *For each* $v \in TM \setminus \{0\}$ *we have:*

(a) $g_{\lambda v} = g_v$*, for all* $\lambda > 0$*;*
(b) $g_v(v, v) = F^2(v)$*;*

(c) $g_v(v,u) = \frac{1}{2}\frac{\partial}{\partial s}F^2(v+su)|_{s=0} = \mathcal{L}(v)u$ *where* $\mathcal{L} : TM\setminus\{0\} \to T^*M\setminus\{0\}$ *is the Legendre transformation;*
(d) $g_v(v,u) \leq F(v)F(u)$, *for all* $u \in T_{\pi(v)}M$.

We can define the *length* of a smooth piecewise curve $\gamma : [a,b] \to M$ as $l_F(\gamma) = \int_a^b F(\gamma'(s))ds$. The *distance* from p to q can be defined as $d(p,q) = \inf_{\gamma\in\Omega_{p,q}} l_F(\gamma)$, where $\Omega_{p,q}$ is the set of curves $\gamma : [0,1] \to M$ joining $p = \gamma(0)$ to $q = \gamma(1)$. Unlike Riemannian geometry, the distance $d(p,q)$ does not need to be equal to the distance $d(q,p)$. But we can still have several important metric geometric concepts from Riemannian geometry, as long as we take into account the orientations of the curves involved in the definition of the distances. For example, instead of talking about a metric ball, now we have to talk about *future balls*, i.e. $B_r^+(p) = \{x \in M \,|\, d(p,x) < r\}$, and *past balls*, i.e. $B_r^-(p) = \{x \in M \,|\, d(x,p) < r\}$.

Since we have a length functional on the space of smooth piecewise oriented curves, we can define a geodesic as an oriented curve that locally minimizes the distance. More precisely a curve $\gamma : [a,b] \to M$ is called *geodesic* if for each $s_0 \in [a,b]$ there exists $\epsilon > 0$ so that $d(\gamma(s_0),\gamma(s)) = \int_{s_0}^s F(\gamma'(t))dt$ where $s \in [s_0, s_0+\epsilon]$. Just like in Riemannian geometry, geodesics can also be seen as critical points of energy functional $\gamma \to \int_a^b F^2(\gamma'(s))ds$ or as curves with zero accelerations with respect to the (Chern) covariant derivative. But before we start to review the concept of Chern connection, let us end this subsection with a concrete important example.

Example 2.13 *(Randers metric).* Let h be a Riemannian metric and $\vec{w}$ be a smooth vector field with $\|\vec{w}\| < 1$, where $\|\vec{w}\| = h(\vec{w},\vec{w})^{1/2}$. We define the Randers metric F with Zermelo data $(h,\vec{w})$ by the intrinsic equation:

$$\|v - F(v)\vec{w}\| = F(v).$$

In other words, $\mathcal{I}_p^F(\epsilon) = \mathcal{I}_p^h(\epsilon) + \epsilon\vec{w}(p)$ where the *indicatrix* $\mathcal{I}_p^F(\epsilon)$ is defined as $\{v \in T_pM \,|\, F(v) = \epsilon\}$. The Randers metric F can also be defined as $F = \alpha + \beta$ where α is a Riemannian norm and β is a 1-form so that $\|\beta\|_\alpha < 1$. There is a bijection between $(h,\vec{w})$ and (α,β), but we will not need it in this paper.

We are interested in two properties of geodesics in Randers manifolds that we formulated as follows:

Proposition 2.14. *Let F be a Randers metric with Zermelo data $(h,\vec{w})$, where $\vec{w}$ is a Killing vector field on M with respect to h. Let γ be a unit speed geodesic with respect to h.*

(a) *Then $t \to \beta(t) = e^{t\vec{w}} \circ \gamma(t)$ is a unit speed geodesic with respect to the Randers metric F.*
(b) *If γ is a unit speed geodesic starting orthogonal to a submanifold L with respect to the Riemannian metric h, i.e. $h(\gamma'(0),v) = 0$ for all $v \in T_{\gamma(0)}L$, then the unit geodesic β (with respect to F) is orthogonal to L with respect to $g_{\beta'}$, i.e., $g_{\beta'(0)}(\beta'(0),v) = 0$ for all $v \in T_{\beta(0)}L$.*

Remark 2.15. There is also an easy way to produce Finsler actions on Randers spaces. An action $\mu : G\times M \to M$ is a Finsler action i.e., $F(d\mu) = F$) on Randers spaces with Zermelo data $(h,\vec{w})$ if and only if the action is isometric (with respect to h) and $\vec{w}$ is G invariant, i.e., $\vec{w} \circ \mu_g = d\mu_g\vec{w}$.

2.2.2. Chern connection and Jacobi fields

Let us now review the concept of Chern connection associated with a Finsler metric F as a family of affine connections.

Proposition 2.16. *Given a vector field $\vec{v}$ without singularities on an open set $U \subset M$, there exists a unique affine connection ∇^v on U (the so called Chern connection) that satisfies the following properties:*

(a) ∇^v *is* torsion-free*, namely,*

$$\nabla^v_{\vec{f}}\vec{g} - \nabla^v_{\vec{g}}\vec{f} = [\vec{f}, \vec{g}]$$

for every vector fields $\vec{f}$ *and* $\vec{g}$ *on* U*,*

(b) ∇^v *is* almost g-compatible*, namely,*

$$\vec{f} \cdot g_v(\vec{g}, \vec{w}) = g_v(\nabla^v_{\vec{f}}\vec{g}, \vec{w}) + g_v(\vec{g}, \nabla^v_{\vec{f}}\vec{w}) + 2\,C_v(\nabla^v_{\vec{f}}\vec{v}, \vec{g}, \vec{w}).$$

Here C_v *is the* Cartan tensor *associated with the Finsler metric defined as:*

$$\begin{aligned} C_v(w_1, w_2, w_3) &:= \frac{1}{2}\frac{\partial}{\partial s} g_{v+sw_1}(w_2, w_3)|_{s=0} \\ &= \frac{1}{4}\frac{\partial^3}{\partial s_3 \partial s_2 \partial s_1} F^2(v + \sum_{i=1}^{3} s_i w_i)|_{s_1=s_2=s_3=0} \end{aligned}$$

for every $p \in M$*,* $v \in T_pM \setminus \{0\}$*, and* $w_1, w_2, w_3 \in T_pM$*.*

It can be checked that the Christoffel symbols of ∇^v only depend on $v = \vec{v}(p)$ at every $p \in M$, and not on the particular extension. Therefore, the Chern connection is an anisotropic connection. Moreover, it is positively homogeneous of degree zero, namely, $\nabla^{\lambda v} = \nabla^v$ for all $v \in TM \setminus \{0\}$ and $\lambda > 0$. One can also prove the following property of Cartan tensor:

$$C_v(v, w_1, w_2) = C_v(w_1, v, w_2) = C_v(w_1, w_2, v) = 0. \tag{2.1}$$

Let $\gamma : I \subset \mathbb{R} \to M$ be a piecewise smooth curve and $t \to \vec{w}(t)$ a vector field without singularities along γ, i.e., $\vec{w}$ is a section of the pullback fiber bundle $\gamma^*(TM)$ over I. By considering the pullback of the Chern connection ∇^w we induce the covariant derivative $\frac{\nabla^w}{dt}$ along γ. In particular we have that $\frac{\nabla^w}{dt}\vec{f}(t) = \nabla^w_{\gamma'}\vec{f}$ when $\vec{f} \in \mathfrak{X}(M)$.

Now we can give an equivalent definition of geodesic. A smooth curve $\gamma : I \subset \mathbb{R} \to M$ is a geodesic if and only if $\frac{\nabla^{\gamma'}}{dt}\gamma'(t) = 0$.

A geodesic can also been seen as the projection of an integral curve of the (Finsler) geodesic spray. In other words, we have a vector field $\vec{\mathrm{g}}$ (*the Finsler geodesic spray*) on $TM - \{0\}$ so that the geodesic γ with initial condition $\gamma'(0) = v_p \in T_pM$ is $\gamma(t) = \pi\big(e^{t\vec{\mathrm{g}}}v_p\big)$, where $\pi : TM \to M$ is the canonical projection. We say that (M, F) is a *complete Finsler manifold*, if the Finsler geodesic spray $\vec{\mathrm{g}}$ is a complete vector field, i.e., its integral curves are defined for all $t \in \mathbb{R}$. The Finsler geodesic spray $\vec{\mathrm{g}}$ has also the interesting property that *its flow preserves a volume form* ω (the so called volume of the *Sasaki metric*) on $TM - \{0\}$, see [10, Propositions 5.4.2, 5.4.3].

When we consider a geodesic variation $t \to \gamma_s(t) = \gamma(s, t)$ of a geodesic γ, then the variational vector field $J(t) = \frac{\partial}{\partial s}\gamma(0, t)$, is called *Jacobi vector field along* γ. It is characterized by solving the differential equation

$$J''(t) + R_{\dot\gamma(t)}(J(t)) = 0. \tag{2.2}$$

Here $J'(t) = \frac{\nabla^{\gamma'}}{dt}J$ and $R_v : T_pM \to T_pM$ is an operator well defined for each $p \in M$ and $v \in T_pM \setminus \{0\}$ called *Jacobi operator*. It can be well defined by properties of isotropic connections; see [3, Section 5].

The *flag curvature* for $v \in TM \setminus \{0\}$ and $w \in T_{\pi(v)}M$ is defined in analogous way to the sectional curvature in Riemannian case.

$$K(v,w) = \frac{g_v(R_v w, v)}{g_v(v,v)g_v(w,w) - g_v(v,w)^2}.$$

Remark 2.17. There is a natural way to produce Finsler spaces with non negative or positive flag curvature. Given a Riemannian manifold (M,h) with nonnegative or positive sectional curvature, the Randers space (M,F) with Zermelo data $(h,\vec{w})$ (where $\vec{w}$ is a Killing field of (M,h)) has nonnegative or positive flag curvature.

The next proposition provides us with a natural relationship between these well-known concepts in Riemannian geometry and their analogues in Finsler geometry.

Proposition 2.18. *Let (M,F) be a complete Finsler manifold and $\vec{v}$ be a geodesic vector field on an open subset $U \subset M$, let $\hat{g} := g_{\vec{v}}$ denote the Riemannian metric on U induced by the fundamental tensor g and let $\widehat{\nabla}$ and $\widehat{R}$ be the Levi-Civita connection and the Jacobi operator of $\hat{g}$, respectively. Then, for any $\vec{f} \in \mathfrak{X}(U)$,*

(a) $\widehat{\nabla}_{\vec{f}}\vec{v} = \nabla^{v}_{\vec{f}}\vec{v}$ *and* $\widehat{\nabla}_{\vec{v}}\vec{f} = \nabla^{v}_{\vec{v}}\vec{f}$,
(b) $\widehat{R}_{\vec{v}}\vec{f} = R_{\vec{v}}\vec{f}$.

As a consequence, the integral curves of $\vec{v}$ are also geodesics of $\widehat{g}$, and the Finslerian Jacobi operator and Jacobi fields along the integral curves of $\vec{v}$ coincide with those of $\widehat{g}$.

We finish this subsection by recalling the concept of L-Jacobi fields.

Definition 2.19. Let L be a submanifold of a complete Finsler manifold (M,F) and $\gamma : [a,b] \to M$ a unit speed geodesic orthogonal to L at $p = \gamma(a)$. We say that a Jacobi field J is a L-Jacobi field if

- $J(a)$ is tangent to L;
- $\mathcal{S}_{\gamma'(a)}J(a) = \tan_{\gamma'(a)}J'(a)$ where $\mathcal{S}_{\gamma'} : T_pL \to T_pL$ is the *shape operator* defined as $\mathcal{S}_{\gamma'}(u) = \tan_{\gamma'(a)}\nabla_u^{\gamma'(a)}\xi$ with ξ an orthogonal vector field along L such that $\xi_p = \gamma'(a)$ and $\tan_{\gamma'(a)}$ is the $g_{\gamma'(a)}$-orthogonal projection onto T_pL.

Remark 2.20. As proved in [3, Proposition 3.5], a Jacobi field J along γ is a L-Jacobi field if and only if it is the variational vector field of a variation of γ by orthogonal geodesics to L.

2.2.3. Finsler submersion

Definition 2.21. A submersion $\rho : (M,F) \to (B,F^\star)$ between Finsler manifolds is a *Finsler submersion* if $d\rho_p(B_p^F(0,1)) = B_{\rho(p)}^{F^\star}(0,1)$, for every $p \in M$, where $B_p^F(0,1)$ and $B_{\rho(p)}^{F^\star}(0,1)$ are the unit balls of the Minkowski spaces (T_pM, F_p) and $(T_{\rho(p)}B, F^\star_{\rho(p)})$ centered at 0, respectively.

The first natural example is to consider a Finsler action $\mu : G \times M \to M$ (i.e., $F(d\mu_g) = F$) where all orbits have the same isotropy type, i.e., the isotropy groups $G_p = \{g \in G \,|\, \mu(g,p) = p\}$ are conjugate to each other. Then the projection $\rho : (M,F) \to (M/G, F^\star)$ is a Finsler submersion where $F^\star$ is the induced Finsler norm on $B = M/G$.

It is also useful to construct Finsler submersions in Randers spaces starting with a Riemannian submersion.

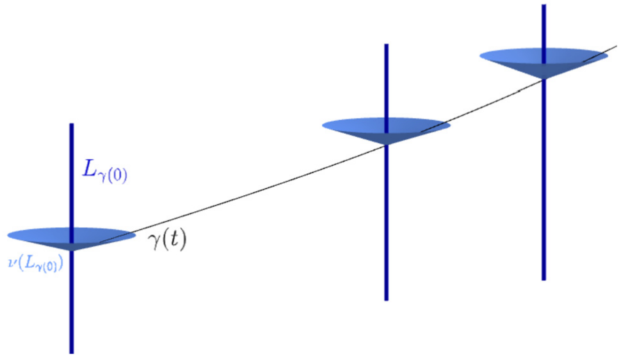

Fig. 1. Figure generated by the software geogebra.org illustrating Lemma 2.22, i.e., a Randers submersion that was produced starting with the trivial Riemannian submersion $\rho : \mathbb{R}^3 \to \mathbb{R}^2$ defined as $\rho(x) = (x_1, x_2)$ and wind $W = (0, 0, \frac{1}{4}\sin^2(x_1) + \frac{1}{4})$. The horizontal unit speed geodesic is $\gamma(t) = (t, 0, \frac{3t}{8} - \frac{\sin(2t)}{16})$. Note that the union of normal vectors to a tangent space of the fiber is a (normal) cone, and no longer a normal subspace as it was in the Riemannian case. As remarked the geodesics of a Finsler submersion are orthogonal to the leaves and hence tangent to the *normal cones* $\nu(L)$.

Lemma 2.22. *Let $\rho : (M, h) \to (B, h^\star)$ be a Riemannian submersion, $\vec{w}^\star$ a vector field on B and $\vec{w}$ a vector field in M that is ρ-related to $\vec{w}^\star$, i.e., $d\rho \circ \vec{w} = \vec{w}^\star \circ \rho$. Then $\rho : (M, R) \to (B, R^\star)$ is a Finsler submersion, where R is the Randers metric with Zermelo data $(h, \vec{w})$ and $R^\star$ is the Randers metric with Zermelo data $(h^\star, \vec{w}^*)$. See Fig. 1.*

Given the Finsler foliation $\mathcal{F} = \{L\}$ with leaves $L = \rho^{-1}(c)$, we say that a geodesic $\gamma : I \subset \mathbb{R} \to M$ is *horizontal* if for each $t \in I$ the vector $\gamma'(t)$ is an orthogonal to the leaves $L \in \mathcal{F}$, i.e., $g_{\gamma'(t)}(\gamma'(t), w) = 0$ for all $w \in T_{\gamma(t)}L$.

In the same way as in Riemannian geometry, in Finsler geometry we have the lift property of geodesics.

Proposition 2.23. *Let $\pi : (M, F) \to (B, F^\star)$ be a Finsler submersion. Then an immersed curve on B is a geodesic if and only if its horizontal lifts are geodesics on M. In particular, the geodesics of $(B, F^\star)$ are precisely the projections of horizontal geodesics of (M, F).*

Once we fix a geodesic vector field, we can reduce the study of Finsler submersions to Riemannian submersions.

Proposition 2.24. *Let $\pi : (M, F) \to (B, F^\star)$ be a Finsler submersion. Let $v^\star$ be a geodesic vector field in some open subset $U^\star$ of B. Then the horizontal lift $\vec{v}$ of v^* is a geodesic vector field on $U = \rho^{-1}(U^\star)$ and the restriction $\rho|_U : (U, g^F_{\vec{v}}) \to (U^\star, g^{F^\star}_{v^*})$ is a Riemannian submersion, where g^F and $g^{F^\star}$ are the fundamental tensors of F and $F^\star$, respectively.*

We finish this subsection by presenting two simple examples illustrating why the compactness hypothesis of Theorem 1.1 is important.

Example 2.25. In this example we present an attainable set and orbit of the set of horizontal unit geodesic vector fields of a Finsler homogeneous analytical submersion on a non compact space, see Fig. 2. Consider the Riemannian submersion $\rho : (\mathbb{R}^2, h_2) \to (\mathbb{R}, h_1)$ where $\rho(x_1, x_2) = x_1$ and h_n is the Euclidean metric on $\mathbb{R}^n$. By Lemma 2.22, taking $\vec{w} = (0, \frac{1}{2})$, we have that $\rho : (\mathbb{R}^2, R) \to (\mathbb{R}, h_1)$ is a Finsler submersion where R is the Randers metric with respect to the Zermelo data $(h_2, \vec{w})$. Let $\mathcal{C} = \{\vec{f_1}, \vec{f_2}\}$ be the set of vector fields where $\vec{f_1} = (1, \frac{1}{2})$ and $\vec{f_2} = (-1, \frac{1}{2})$. The integral curves of these vector fields are horizontal geodesics of the Finsler submersion $\rho : (\mathbb{R}^2, R) \to (\mathbb{R}, h_1)$. It is easy to see that $\mathcal{A}_{(0,0)}(\mathcal{C})$ is a cone with its interior. More precisely $\mathcal{A}_{(0,0)}(\mathcal{C}) = \{x \in \mathbb{R}^2 \,|\, \frac{1}{2}x_1 \le x_2, 0 \le x_1\} \cup \{x \in \mathbb{R}^2 \,|\, -\frac{1}{2}x_1 \le x_2, x_1 \le 0\}$. Also it is clear that $\mathcal{O}\big((0,0)\big) = \mathbb{R}^2$. In particular $\mathcal{O}\big((0,0)\big) \ne \mathcal{A}_{(0,0)}(\mathcal{C})$.

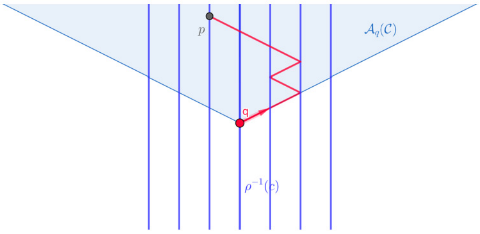

Fig. 2. Figure generated by the software geogebra.org illustrating Example 2.25.

Example 2.26. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$ be the canonical projection of the Euclidean plane onto the torus and the Finsler submersion $\rho : (\mathbb{R}^2, R) \to (\mathbb{R}, h_1)$ defined in Example 2.25. Since the fibers of the submersion ρ and the vector field $\vec{w}$ are invariant by the action of $\mathbb{Z} \times \mathbb{Z}$, the Finsler submersion ρ projects to a Finsler submersion $\rho^\star : (\mathbb{T}^2, R^\star) \to (\mathbb{S}^1, h_1)$. Here $R^\star$ is the Randers metric with Zermelo data $(h_2, w^\star)$ where the h_2 is flat metric and the wind $w^\star$ is ρ-related with $\vec{w}$, i.e., $w^\star \circ \pi = d\pi \circ \vec{w}$. Define $f_i^\star$ to be π-related to $\vec{f}_i$. The integral curves of $f_i^\star$ are horizontal geodesics of $\rho^\star$. Set $\mathcal{C}^\star = \{f_1^\star, f_2^\star\}$ and $p^\star = \pi((0,0))$. Then it is not difficult to check that $\mathcal{A}_{p^\star}(\mathcal{C}^\star) = \mathcal{O}(p^\star)$.

3. Proof of item (a) of Theorem 1.1

Let $N \subset T^1M$ be the union of unit cone bundle of the fibers of ρ, i.e., $N := \cup_{x \in M} \nu_x^1(L_x)$ for $L_x = \rho^{-1}(\rho(x))$. It follows from Alvarez Paiva and Duran [8] that N is a compact embedded submanifold of the unit bundle T^1M and that the diagram below commutes

$$\begin{array}{ccc} N & \xrightarrow{\ \rho_N\ } & T^1B \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_B} \\ M & \xrightarrow[\ \rho\ ]{} & B \end{array} \tag{3.1}$$

where $\rho_N = d\rho|_N$ and π_M and π_B are the canonical projections. Also note that N is invariant by the geodesic flow $e^{t\vec{\mathfrak{g}}}$ and

$$\rho_N \circ e^{t\vec{\mathfrak{g}}} = e^{t\vec{\mathfrak{b}}} \circ \rho_N \tag{3.2}$$

where $e^{t\vec{\mathfrak{b}}}$ is the geodesic flow in T^1B.

Remark 3.1. Note that the isometric action $\mu : G \times M \to M$ induces an action $\tilde{\mu} : G \times N \to N$ as $\tilde{\mu}_g = (\mu_g)_*$ and the orbits of the action induce the leaves of the foliation $\widetilde{\mathcal{F}} := \{\widetilde{L}\}$ where $\widetilde{L} = \rho_N^{-1}(c)$.

Lemma 3.2. *The Finsler geodesic spray $\vec{\mathfrak{g}}$ restricted to N is Poisson stable.*

Proof. In order to prove that the flow is Poisson stable, it suffices to check that

$$|e^{t\vec{\mathfrak{g}}}(W)| = |W|, \quad \forall t > 0, \tag{3.3}$$

where W is any given proper relative compact neighborhood and $|\cdot|$ is a fixed volume on N that will be constructed below, recall Proposition 2.7.

The first step in our construction is to define a metric on the fibers of $\rho : M \to B = M/G$ so that, for each basic vector field ξ, the end point map $\eta_\xi : G(x) \to G(y)$, defined as $\eta_\xi(x) = \exp_x(\xi)$, turns to be an isometry.

Since all orbits are principal, the slice theorem implies that the map $x \mapsto \mathfrak{g}_x \subset Gk(\mathfrak{g})$ is smooth (where $Gk(\mathfrak{g})$ is the Grassmannian of $\mathfrak{g}$). For a given metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, we can find a subspace V_x orthogonal to $\mathfrak{g}_x$, i.e., $\mathfrak{g} = V_x \oplus \mathfrak{g}_x$, where $\mathfrak{g}$ and $\mathfrak{g}_x$ are the Lie algebra of G and the isotropy group G_x respectively. Now we define the metric $\tilde{g}$ along the orbits that transform the isomorphism $d\mu_x : (V_x, \langle \cdot, \cdot \rangle) \to (T_x G(x), \tilde{g}_x)$ into an isometry.

Since the isotropy groups along (minimal) horizontal geodesics are the same, we have that $V_x = V_y$, where $y = \eta_\xi(x)$. Note that $d\eta_\xi \vec{v}(x) = \vec{v}(y)$ where $\vec{v}$ is the vector field along the orbits defined as $\vec{v}(x) = d\mu_x v$ for $v \in V = V_x = V_y$. These facts allow us to conclude that the map $\eta_\xi : G(x) \to G(y)$ is an isometry.

Now we can define a volume form ω_G (with respect to the metric $\tilde{g}$) along the fibers of $\rho : M \to B = M/G$. Note that this form is invariant by the end point map, i.e., $\eta_\xi^* \omega_G = \omega_G$. The metric $\tilde{g}$ as well the volume form ω_G can also be defined on the fibers $\{\widetilde{L}\}$ of $\rho_N : N \to T^1 B$ and we will use the same notation. Note that if $e^{t\vec{\mathfrak{g}}} : \widetilde{L}_x \to \widetilde{L}_y$, then $(e^{t\vec{\mathfrak{g}}})^* \omega_G = \omega_G$. By another abuse of notation, consider ω_G an extension of the previous ω_G to a k-form in N.

We can define the volume form as $\omega = \omega_G \wedge \rho_N^* \omega_B$ where ω_B is the volume form with respect to the Sasaki metric of $T^1 B$. Recall that $e^{t\vec{\mathfrak{b}}}$ preserves the volume form for the Sasaki metric of $T^1 B$ and $(e^{t\vec{g}})^* \omega_G|_{\widetilde{L}_x} = \omega_G|_{\widetilde{L}_y}$. These facts together with the fact that $\rho_N^* \omega_B$ vanishes vectors tangent to fibers allow one to check that:

$$(e^{t\vec{\mathfrak{g}}})^* \omega = \omega. \tag{3.4}$$

Defining $|W| = \int_W \omega$, we conclude that Eq. (3.4) implies Eq. (3.3) and this concludes the proof of the lemma. □

Remark 3.3. Note that the fact that Finsler geodesic spray preserves volume on TM does not directly imply that its restriction to a submanifold of TM preserves volume of this submanifold. This is one of the reasons why we are assuming that the submersion $\rho : M \to B$ is homogeneous.

Now we are going to define a set of vector fields $\widetilde{\mathcal{C}}$ on N that will be related to our original set of vector fields $\mathcal{C}$ on M as follows:

(i) $\pi_M\big(\widetilde{\mathcal{A}}_{\tilde{q}}(\widetilde{\mathcal{C}})\big) = \mathcal{A}_q(\mathcal{C})$,
(ii) $\pi_M\big(\widetilde{\mathcal{O}}(\tilde{q})\big) = \mathcal{O}(q)$,

where $\pi_M(\tilde{q}) = q$. Therefore, once we have proved that $\widetilde{\mathcal{A}}_{\tilde{q}}(\mathcal{C}) = \widetilde{\mathcal{O}}(\tilde{q})$ (see Eq. (3.5) below) we will be able to conclude that $\mathcal{A}_q(\mathcal{C}) = \mathcal{O}(q)$ and hence to finish the proof of item (a) of Theorem 1.1.

Consider a set of vector fields $\mathcal{C}_1 = \{\vec{f}_u\}$ with the following properties: $\vec{f}_u$ spam the tangent spaces of the fibers of $\pi_M : N \to M$, $\mathcal{C}_1$ is symmetric (i.e. if $\vec{f}_u \in \mathcal{C}_1$ then $-\vec{f}_u \in \mathcal{C}_1$) and $\vec{f}_u$ are μ_g-related, i.e., $\vec{f}_u \circ \tilde{\mu}_g = d\tilde{\mu}_g \vec{f}_u$. Now we complete $\mathcal{C}_1$ with the geodesic spray (restricted to N), i.e., $\widetilde{\mathcal{C}} = \{\vec{\mathfrak{g}}\} \cup \mathcal{C}_1$. Note that the projection of the integral curves of $\vec{f}_0 = \vec{\mathfrak{g}}$ corresponds to the horizontal geodesics and the projection of the integral curves of $\vec{f}_u$ $(u \neq 0)$ measure how it breaks, and in particular $\pi_M(\gamma_2 * \delta_u * \gamma_1)$ is a broken horizontal unit speed geodesic, where γ_i is a integral line of $\vec{\mathfrak{g}}$ and δ_u is an integral line of $\vec{f}_u$. The attainable set and the orbit of the family $\widetilde{\mathcal{C}}$ through $\tilde{q}_0$ are denoted by $\widetilde{\mathcal{O}}(\tilde{q}_0)$ and $\widetilde{\mathcal{A}}_{\tilde{q}_0}(\widetilde{\mathcal{C}})$. Using this one can check properties (i) and (ii) stated above.

Lemma 3.4. *Consider $\widetilde{\mathcal{O}}(\tilde{q})$ the orbit of the family $\widetilde{\mathcal{C}}$ through $\tilde{q}$ defined above. Then*

(a) Each orbit $\widetilde{\mathcal{O}}(\tilde{q})$ meet all the fibers of ρ_N.
(b) The orbits of $\{\widetilde{\mathcal{O}}(\tilde{q})\}$ have the same dimension (i.e., $\{\widetilde{\mathcal{O}}(\tilde{q})\}$ is a regular foliation).
(c) If the orbits $\mathcal{O}(q)$ in M are embedded then the orbits $\widetilde{\mathcal{O}}(\tilde{q})$ in N are embedded as well.
(d) If the orbits $\widetilde{\mathcal{O}}(\tilde{q})$ in N are embedded, then $\vec{\mathrm{g}}$ restricts to each orbit is Poisson stable.

Proof. *Item (a)* Fix a point $\tilde{q}_0 \in N$ and consider $\tilde{q}_1 \in N$. Let $\gamma_B : [0, r] \to B$ be a piecewise broken unit geodesic so that $\gamma_B'(0) = \rho_N(\tilde{q}_0)$ and $\gamma_B'(r) = \rho_N(\tilde{q}_1)$. By lifting horizontally via ρ and then lifting via π_M (see (3.1)) we define $\gamma_N : [0, r] \to N$ as the lift of γ_B with $\gamma_N'(0) = \tilde{q}_0$. Set $\gamma_N'(r) = \tilde{q}_2$. Note that $\rho_N(\tilde{q}_2) = \rho_N(\tilde{q}_1)$ and hence that $\tilde{q}_2 \in \widetilde{L}_{\tilde{q}_1}$. Since $\gamma_N \subset \widetilde{\mathcal{O}}(\tilde{q}_0)$, we have concluded that $\tilde{q}_2 \in \widetilde{\mathcal{O}}(\tilde{q}_0) \cap \widetilde{L}_{\tilde{q}_1}$ and this finishes the proof of item (a).

Item (b) Recall that the action $\mu : G \times M \to M$ induces an action $\tilde{\mu} : G \times TM \to TM$ as $\tilde{\mu}_g = (\mu_g)_*$ and the orbits of the action $\tilde{\mu}$ induce the leaves of $\widetilde{\mathcal{F}}$. For a fix leaf $\widetilde{L}_{\tilde{q}}$, note that $\widetilde{\mu}_g(\widetilde{\mathcal{O}}(\tilde{q})) = \widetilde{\mathcal{O}}(\widetilde{\mu}_g(\tilde{q}))$. Since $\widetilde{\mu}_g$ is a diffeomorphism, the orbits that meet $\widetilde{L}_{\tilde{q}}$ have the same dimension. On the other hand, it follows from item (a) that all orbits meet $\widetilde{\mathcal{F}}$. Therefore all orbits have the same dimension.

Item (c) Since π_M is a submersion (consequently transverse to $\mathcal{O}(p)$) it suffices to prove that $\pi_M^{-1}(\mathcal{O}(p)) = \widetilde{\mathcal{O}}(u_p)$, for all $u_p \in N$. The inclusion $\pi_M^{-1}(\mathcal{O}(p)) \supset \widetilde{\mathcal{O}}(u_p)$ follows immediately from the fact that $\pi_M(\widetilde{\mathcal{O}}(u_p)) = \mathcal{O}(p)$.

In order to prove that $\pi_M^{-1}(\mathcal{O}(p)) \subset \widetilde{\mathcal{O}}(u_p)$ consider $v_q \in \pi_M^{-1}(\mathcal{O}(p))$. Then $q = \pi_M(v_q) \in \mathcal{O}(p)$ which means that p and q are linked by a broken curve $\gamma = \gamma_n * \cdots * \gamma_1$ where each γ_i segment is either unit horizontal geodesic or a reverse of a unit horizontal geodesic.

Set $\widetilde{\gamma}_i(t) := (\gamma_i'(t))_{\gamma_i(t)} \in T_{\gamma_i(t)}M$ if γ_i a unit segment of geodesic and set $\widetilde{\gamma}_i(t) := (-\gamma_i'(t))_{\gamma_i(t)} \in T_{\gamma_i(t)}M$ if γ_i is a reverse segment of geodesic. Let δ_1 be an integral line of $\mathcal{C}_1$ connecting u_p with $\widetilde{\gamma}_1(0)$, δ_i be an integral line of $\mathcal{C}_1$ joining $\widetilde{\gamma}_{i-1}(r_{i-1})$ with $\widetilde{\gamma}_i(r_i)$ and δ_{n+1} be an integral line of $\mathcal{C}_1$ joining $\widetilde{\gamma}_n(r_n)$ with v_q. Then $\delta_{n+1} * \widetilde{\gamma}_n * \delta_n * \widetilde{\gamma}_{n-1} * \delta_{n-1} * \cdots * \widetilde{\gamma}_1 * \delta_1$ is a broken curve connecting u_p with v_q, where each segment is either a integral line of $\widetilde{\mathcal{C}}$ or a reverse of a integral line of $\widetilde{\mathcal{C}}$ which means that $v_q \in \widetilde{\mathcal{O}}(u_p)$.

Item (d) For a fixed $\tilde{q}_0 \in N$ consider a small relatively compact trivial $\widetilde{\mathcal{F}}$-neighborhood V_0 of $\tilde{q}_0$ so that $V_0 \cap \widetilde{\mathcal{O}}(\tilde{q}_0)$ has only one connected component. We want to check that for each t_0 there exists $t_1 > t_0$ and a point $\tilde{x} \in V_0 \cap \widetilde{\mathcal{O}}(\tilde{q}_0)$ so that $\widetilde{x}_1 = e^{t_1 \vec{\mathrm{g}}}(\tilde{x}) \in V_0 \cap \widetilde{\mathcal{O}}(\tilde{q}_0)$.

We claim that there exists a relatively compact neighborhood $V_1 \subset V_0$ and a neighborhood W of $e \in G$ so that:

- if $g \in W$ then $g^{-1} \in W$ and $\mu_{g^{-1}}(V_1) \subset V_0$ and $\mu_g(V_1) \subset V_0$;
- if $\tilde{y} \in V_1$ then there exists $\tilde{x} \in \widetilde{\mathcal{O}}(\tilde{q}_0) \cap V_1$ so that $\mu_g(\tilde{x}) = \tilde{y}$, with $g \in W$.

By Lemma 3.2, for each t_0, there exists $\tilde{y} \in V_1$ and $t_1 > t_0$ so that $e^{t_1 \vec{\mathrm{g}}}(\tilde{y}) = \tilde{y}_1 \in V_1$. Consider $\tilde{x} \in V_1 \cap \widetilde{\mathcal{O}}(\tilde{q}_0)$ so that $\tilde{y} = \mu_g(\tilde{x})$. Since $\mu_g \circ e^{t_1 \vec{\mathrm{g}}} = e^{t_1 \vec{\mathrm{g}}} \circ \mu_g$, we have $\tilde{y}_1 = e^{t_1 \vec{\mathrm{g}}}\big(\mu_g(\tilde{x})\big) = \mu_g\big(e^{t_1 \vec{\mathrm{g}}}(\tilde{x})\big)$ and hence $\tilde{x}_1 := \mu_{g^{-1}}(\tilde{y}_1) = e^{t_1 \vec{\mathrm{g}}}(\tilde{x})$. Once $\mu_{g^{-1}}(V_1) \subset V_0$ and $V_0 \cap \widetilde{\mathcal{O}}(\tilde{q}_0)$ has only one connected component, we infer that $\tilde{x}_1 \in V_0 \cap \widetilde{\mathcal{O}}(\tilde{q}_0)$ as we wanted to prove. □

We can now end the proof of item (a) of Theorem 1.1. Item (d) of Lemma 3.4 implies that $\vec{\mathrm{g}}$ restricts to each orbit is Poisson stable. From Proposition 2.9 $-\vec{\mathrm{g}}$ is compatible with $\widetilde{\mathcal{C}}$, i.e. $\widetilde{\mathcal{A}}_{\tilde{q}}(\widetilde{\mathcal{C}})$ is dense in $\widetilde{\mathcal{A}}_{\tilde{q}}(\widetilde{\mathcal{C}} \cup \{-\vec{\mathrm{g}}\}) = \widetilde{\mathcal{O}}(\tilde{q})$. Therefore, it follows from Proposition 2.10 that

$$\widetilde{\mathcal{A}}_{\tilde{q}}(\widetilde{\mathcal{C}}) = \widetilde{\mathcal{O}}(\tilde{q}). \tag{3.5}$$

The above equation finishes the proof as remarked before.

4. Proof of Proposition 1.2 and item (b) of Theorem 1.1

Let us start by recalling the definition of singular Finsler foliation (SFF for short), a class of singular foliation that includes among other examples, the partition of M into orbits of a Finsler action; for properties and more examples of SFF see [3] and see [2].

Definition 4.1 *(SFF).* A partition $\mathcal{F} = \{L\}$ on a complete Finsler manifold (M, F) is called a *singular Finsler foliation* if it satisfies the following two conditions:

(a) $\mathcal{F}$ is a *singular foliation*, i.e., for each $p \in M$, each basis $\{X_i\}$ of the tangent space T_pL_p of the leaf L_p through p, can be extended to vector fields $\{\vec{X}_i\}$ linearly independent, tangent to the leaves of $\mathcal{F}$ near p.

(b) $\mathcal{F}$ is *Finsler*, in other words, if a geodesic $\gamma : (-\epsilon, \epsilon) \to M$ is orthogonal to the leaf $L_{\gamma(0)}$ (i.e. $g_{\gamma'(0)}(\gamma'(0), v) = 0$ for each $v \in T_{\gamma(0)}L$) then γ is horizontal, i.e., orthogonal to each leaf it meets.

We also need to present a result that is a direct consequence of Lemma 5.11, a version of Wilking's lemma for Finsler geometry.

Proposition 4.2. *Let (M, F) be a complete Finsler manifold with non negative flag curvature along geodesic $\gamma : \mathbb{R} \to M$ orthogonal (at its initial point) to a submanifold $L \subset M$. Denote $\mathcal{J}_\gamma^L$ the set of all L-Jacobi fields along γ. Then*

$$\mathcal{J}_\gamma^L = \operatorname{span}_{\mathbb{R}}\{J \in \mathcal{J}_\gamma^L \mid J(t) = 0 \text{ for some } t \in \mathbb{R}\} \oplus \{J \in \mathcal{J}_\gamma^L \mid J \text{ is parallel}\}.$$

As explained in the introduction, item (b) of Theorem 1.1 follows direct from item (a) of Theorem 1.1 and Proposition 1.2. Therefore let us prove this proposition in this section.

Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic orthogonal to a regular leaf L at $\gamma(0)$ (i.e., γ is an horizontal geodesic). First we want to check that the first summand of the decomposition presented in Proposition 4.2 is tangent to the orbit $\mathcal{O}(\gamma(0))$. To prove this it suffices to prove the next lemma.

Lemma 4.3. *If a Jacobi field $J \in \mathcal{J}_\gamma^L$ has zero at t_0 (i.e., $J(t_0) = 0$) then it is tangent to the orbit $\mathcal{O}(\gamma(0))$.*

Proof. Let $t \to \gamma_s(t) = \gamma(s, t)$ be a variation of horizontal unit geodesics orthogonal to L_p with $p = \gamma(0, 0) = \gamma(0)$ and so that $J(t) = \frac{\partial}{\partial s}\gamma(0, t)$. Consider a basis $\{X_i\}$ of $T_{\gamma(t_0)}L_{\gamma(t_0)}$. It follows from item (a) of Definition 4.1 that these vectors can be extended to vector fields $\{\vec{X}_i\}$ that are linearly independent. From item (b) of Definition 4.1, we have that $0 = g_{\gamma_s'}(\vec{X}_i, \gamma_s')$. By differentiating this equation and taking into account item (b) of Proposition 2.16, we infer:

$$\begin{aligned}
0 &= \frac{\partial}{\partial s} g_{\gamma_s'}(\vec{X}_i, \gamma_s'(t)) \\
&= g_{\gamma_s'}\Big(\frac{\nabla^{\gamma_s'}}{\partial s}\vec{X}_i, \gamma_s'(t)\Big) \\
&+ g_{\gamma_s'}\Big(\vec{X}_i, \frac{\nabla^{\gamma_s'}}{\partial s}\frac{\partial}{\partial t}\gamma_s(t)\Big) \\
&+ 2C_{\gamma_s'}\Big(\frac{\nabla^{\gamma_s'}}{\partial s}\gamma_s'(t), \vec{X}_i, \gamma_s'(t)\Big)
\end{aligned}$$

The above equation, the fact that $\frac{\nabla^{\gamma_s'}}{\partial s}\vec{X}_i|_{s=0,t=t_0} = \nabla^{\gamma_0'}_{J(t_0)}\vec{X}_i = \nabla^{\gamma_0'}_0\vec{X}_i = 0$ and Eq. (2.1) allow us to conclude that

$$0 = g_{\gamma_0'}\left(\vec{X}_i, \frac{\nabla^{\gamma_0'}}{\partial t}\frac{\partial}{\partial s}\gamma_s(t_0)|_{s=0}\right) = g_{\gamma_0'}(X_i, \frac{\nabla^{\gamma_0'}}{\partial t}J(t_0)). \tag{4.1}$$

Eq. (4.1) implies that

$$J'(t_0) \in \mathcal{H}(t_0) \, \text{and} \, J(t_0) = 0. \tag{4.2}$$

Here $\mathcal{H}(t_0) = \{w \in T_{\gamma(t_0)}M|, g_{\gamma'(t_0)}(w, X_i(\gamma(t_0)) = 0, \forall i\}$.

We claim that *the normal cone $\nu_{\gamma(t_0)}(L_{\gamma(t_0)})$ is tangent to $\mathcal{H}(t_0)$*. In order to check this claim, consider a curve $s \to v(s)$ with $v(0) = \gamma'(t_0)$ contained in the unit normal cone, i.e., $g_{v(s)}(v(s), X_i(\gamma(t_0))) = 0$, $\forall i$. By differentiating this equation with respect to s and taking into account item (b) of Proposition 2.16, we infer that $g_{\gamma'(t_0)}(v'(0), X_i(\gamma(t_0))) = 0$, $\forall i$, i.e., that $v'(0)$ is tangent to $\mathcal{H}(t_0)$. An argument comparing dimensions allows one to conclude the proof of the claim, see also proof of [3, Lemma 2.9].

The claim and Eq. (4.2) imply the existence of a variation of geodesics $t \to f(s,t)$ so that

- $t \to f(0,t) = \gamma(t)$,
- $t \to f(s,t)$ are geodesics orthogonal to $L_{\gamma(t_0)}$ i.e., contained in $\mathcal{O}(\gamma(t_0))$,
- $f(s,t_0) = f(0,t_0) = \gamma(t_0)$ and $J(t) = \frac{\partial}{\partial s}f(0,t)$.

In fact we can define $f(s,t) = \pi\left(e^{(t-t_0)\vec{\mathrm{g}}}v(s)\right)$ where $\pi : TM \to M$ is the canonical projection and $s \to v(s)$ is a curve contained in unit cone $\nu^1_{\gamma(t_0)}(L_{\gamma(t_0)})$ with $v'(0) = J'(t_0)$ and $v(0) = \gamma'(t_0)$.

Since $t \to f(s,t)$ are geodesics contained in $\mathcal{O}(\gamma(t_0))$ and $\gamma(0) \in \mathcal{O}(\gamma(t_0))$, we conclude that the variation $t \to f(s,t)$ is contained in $\mathcal{O}(\gamma(0))$ and hence that $t \to J(t) = \frac{\partial}{\partial s}f(0,t)$ is tangent to $\mathcal{O}(\gamma(0))$ what finishes the proof. □

Let us now check that codimension of the dual leaf is zero. Assume by contradiction that there exists $v \in L_{\gamma(0)}$ orthogonal to $\mathcal{O}(\gamma(0))$, where γ is an horizontal geodesic with $\gamma(0) = q$ with $K(q) > 0$ for $q \in L_{q_0}$. Consider a $L_{\gamma(0)}$-Jacobi field J, so that $J(0) = v$. Since we have proved above that the first summand of the decomposition presented in Proposition 4.2 is tangent to the orbit $\mathcal{O}(\gamma(0))$, we have that J can not be contained in this summand. Hence, by Proposition 4.2, J must be a non trivial parallel Jacobi vector field, that implies that the curvature can not be positive, what contradicts our hypothesis that $K(q) > 0$.

Since we have proved that $\mathcal{O}(q)$ has codimension zero for each $q \in L_{q_0}$ and each point $x \in M$ is contained in an orbit $\mathcal{O}(q)$ (for $q \in L_{q_0}$), we conclude that $\mathcal{O}(q_0) = M$.

It follows from item (a) of Theorem 1.1 that $M = \mathcal{O}(q_0) = \mathcal{O}(q) = \mathcal{A}_q(\mathcal{C})$. This concludes the proof of Proposition 1.2 and hence the proof of item (b) of Theorem 1.1.

5. Wilking's transverse Jacobi fields

We reproduce here, in more general context, the construction of transverse Jacobi fields presented in [13], in [7] and in [9] in order to obtain a Finslerian version of the Corollary 10 in [13], i.e., Proposition 4.2.

5.1. Jacobi triples and Jacobi equation

A *Jacobi triple* (E, D, R) is composed by

- Euclidean vector field E (total space) over a open interval $I \subset \mathbb{R}$ (with rank n);
- a covariant derivative $D : \Gamma(E) \to \Gamma(E)$ compatible with the fiberwise metric of E;
- a self-adjoint $C^\infty(I)$-homomorphism $R : \Gamma(E) \to \Gamma(E)$.

Presented in this way, this definition seems a little artificial. Indeed, this is an algebraic approach whose intention is to condense the relevant data and properties of Jacobi fields that will be useful throughout section 5.

Given a Jacobi triple (E, D, R), the kernel of the second order differential operator $D^2 + R$ will be called the space of (E, D, R)-Jacobi fields (or simply *Jacobi fields*) and will be denoted by $\mathcal{J}(E, D, R)$ (or simply by $\mathcal{J}$). Since $D^2 + R$ is a linear operator, for each $t \in I$, the map $J \longmapsto (J(t), DJ(t))$ is an isomorphism between $\mathcal{J}$ and $E_t \oplus E_t$. In particular, $\dim(\mathcal{J}) = 2\,\mathrm{rank}\,(E) = 2n$.

The space of Jacobi fields $\mathcal{J}$ inherits a canonical symplectic form ω given by

$$\omega(J_1, J_2) := \langle DJ_1, J_2\rangle - \langle J_1, DJ_2\rangle\,.$$

Note that the right term of this definition is in fact independent of the t parameter. Precisely ω is the pullback of the canonical form by the isomorphism $J \longmapsto (J(t), DJ(t))$. As usual, the space of Lagrangian subspaces of $(\mathcal{J}, \omega)$ (the Grassmannian Lagrangian of $(\mathcal{J}, \omega)$) will be denoted by $\Lambda(\mathcal{J})$, i.e.

$$\Lambda(\mathcal{J}) := \{\mathcal{L} \subset \mathcal{J} : \mathcal{L} \text{ is a Lagrangian subspace of } \mathcal{J}\}. \tag{5.1}$$

Finally we establish some useful notation. Given $\mathcal{I} \subset \mathcal{J}$ a vector subspace of Jacobi fields for each $t \in I$ it will be denoted

$$\mathcal{I}(t) := \{J(t) \in E_t : J \in \mathcal{I}\} \qquad \text{and} \qquad \mathcal{I}^0_t := \{J \in \mathcal{I} : J(t) = 0\}.$$

Following this notation we point out that $\mathcal{I}^0_t$ is isomorphic to $D(\mathcal{I}^0_t)(t) = \{DJ(t) : J \in \mathcal{I}^0_t\}$.

5.2. Illustrative examples

Looking for a consolidation of our algebraic approach we present some geometrical examples of subspaces of Jacobi fields in an increasing rate of complexity. The focus is the subspaces determined by the symplectic structure (i.e. isotropic and Lagrangian subspaces). Some future notation will be presented as well.

Example 5.1 *(Finslerian Jacobi fields).* Let M be a Finsler manifold with fundamental tensor g and γ a geodesic segment. Observe that $(\gamma^*TM, g_{\gamma'(\cdot)})$ is a Euclidian vector bundle over I. Denote by $D^{\gamma'}_{\gamma}$ the Chern covariant derivative along γ and $R^{\gamma'}$ the Jacobi operator along γ. Then $(\gamma^*TM, D^{\gamma'}, R_{\gamma'})$ is a Jacobi triple.

The set of Jacobi fields associated with $(\gamma^*TM, D^{\gamma'}, R_{\gamma'})$ will be denoted by $\mathcal{J}_\gamma$.

In the next two examples we are going to use the notation established in the previous example.

Example 5.2 *(L-Jacobi fields).* Let M^n be a Finsler manifold, $L \subset M$ a immersed submanifold and $\gamma : [a, b] \subset \mathbb{R} \longrightarrow M$ a geodesic segment such that $\gamma(a) \in L$ and $\gamma'(a)$ is $g_{\gamma'(a)}$-orthogonal to L. Denote $\mathrm{pr}_L : T_{\gamma(a)}M \to T_{\gamma(a)}L$ the canonical projection with respect to the $g_{\gamma'}$-orthogonal decomposition of $T_{\gamma(a)}M$.

The set o L-Jacobi fields is defined by

$$\mathcal{J}^L_\gamma := \{J \in \mathcal{J}_\gamma : J(a) \in T_{\gamma(a)}L \text{ and } \mathrm{pr}_L(D^{\gamma'}_\gamma J(a)) = S_{\gamma'(a)}(J(a))\},$$

where $S_{\gamma'(a)}$ is the shape operator of L in the direction $\gamma'(a)$. As we previously discuss, this is precisely the Jacobi fields obtained by variations of γ through geodesics starting perpendicular to L. An advantage in this presentation is that will became more easily to see that $\mathcal{J}^L_\gamma$ is a Lagrangian subspace of Jacobi fields. Clearly the self-adjointness of $S_{\gamma'(a)}$ guarantees that $\mathcal{J}^L_\gamma$ is isotropic, i.e.,

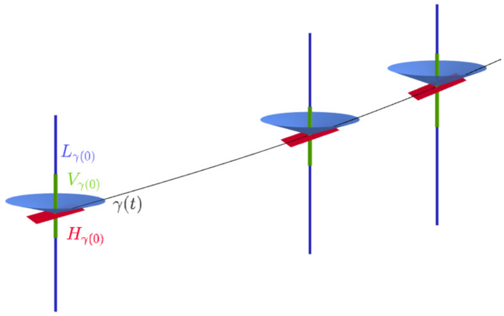

Fig. 3. Figure generated by the software geogebra.org illustrating a submersion on Randers space, with horizontal geodesic γ, horizontal bundle H_t and the vertical bundle V_t of Finsler submersion, see Example 5.3.

$$\begin{aligned}\omega(J_1, J_2) &= \langle D_\gamma^{\gamma'}(J_1), J_2\rangle - \langle J_1, D_\gamma^{\gamma'}(J_2)\rangle \\ &= \langle -S_{\gamma'(a)}(J_1(a)), J_2(a)\rangle - \langle J_1(a), -S_{\gamma'(a)}(J_2(a))\rangle \\ &= 0\end{aligned}$$

where $J_1, J_2 \in \mathcal{J}_\gamma^L$. The dimension of $\mathcal{J}_\gamma^L$ is determined by the linearly independent choices for the initial conditions of its Jacobi fields which implies that $\dim(\mathcal{J}_\gamma^L) = \dim(T_{\gamma(a)}L) + \dim((T_{\gamma(a)}L)^\omega) = \frac{1}{2}\dim(\mathcal{J}_\gamma)$.

Example 5.3 *(Finsler submersions).* Let $\pi : M^{m+k} \longrightarrow B^k$ be a Finsler submersion and $\gamma : [a, b] \subset \mathbb{R} \longrightarrow M$ a horizontal geodesic; see Fig. 3. Along γ it is possible to consider a horizontal bundle H by $g_{\gamma'}$-orthogonal complement of the vertical bundle $V := \gamma^* \operatorname{Ker}(d\pi)$, which will allow us to define operators analogous to the O'Neill tensors in Riemannian submersions and thus be able to work with holonomy type of Jacobi fields and projectable Jacobi fields just like in the Riemannian case. More precisely, we consider $S_{\gamma'} : \Gamma(V) \to \Gamma(V)$ the shape operator of the fibers in the direction γ', we define $\mathbb{A}_{\gamma'} : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$ by $\mathbb{A}_{\gamma'}(X) := (D_\gamma^{\gamma'} X^V)^H + (D_\gamma^{\gamma'} X^H)^V$ and denote $A_{\gamma'} := \mathbb{A}_{\gamma'}|_{\Gamma(H)}$.

The set of holonomy type Jacobi fields (along γ) is defined by

$$\mathcal{J}_\gamma^{\text{hol}} := \{J \in \mathcal{J}_\gamma : J(a) \in V_{\gamma(a)} \text{ and } D_\gamma^{\gamma'} J(a) = -(S_{\gamma'(a)} + A^*_{\gamma'(a)})(J(a))\}$$

and this is an example of isotropic subspace of $\mathcal{J}_\gamma$. This is simple to verify since the shape operator is self-adjoint and the holonomy type Jacobi fields are vertical in its initial point. In fact they are everywhere vertical. Additionally, once a Jacobi field is determined by its initial conditions it is simple to conclude that $\dim(\mathcal{J}_\gamma^{\text{hol}}) = \operatorname{rank}(V) = k$.

Like in the case of L-Jacobi fields it is more suitable for computations to define $\mathcal{J}_\gamma^{\text{hol}}$ algebraically and postpone its geometrical meaning. In this case they can be obtained by variations of γ through horizontal geodesics that are horizontal lifts of the geodesic $\pi \circ \gamma$.

Another remarkable space of Jacobi field associated with a Finsler submersion is the set of projectable Jacobi fields which is defined by

$$\mathcal{J}_\gamma^{\text{proj}} := \{J \in \mathcal{J}_\gamma : D_\gamma^{\gamma'} J^V = -S_{\gamma'}(J^V) - A_{\gamma'}(J^H)\}$$

and this is an example of coisotropic subspace of $\mathcal{J}_\gamma$. More precisely this is the symplectic orthogonal of $\mathcal{J}_\gamma^{\text{hol}}$. To see this, we first observe that $\mathcal{J}_\gamma^{\text{proj}} \subset (\mathcal{J}_\gamma^{\text{hol}})^\omega$ which is a direct consequence of the subsequent computation

$$\begin{aligned}\omega(J_1, J_2) &= \langle D_\gamma^{\gamma'} J_1, J_2\rangle - \langle J_1, D_\gamma^{\gamma'} J_2\rangle \\ &= \langle -S_{\gamma'(a)}(J_1(a)), J_2^V(a)\rangle + \langle -A^*_{\gamma'(a)}(J_1(a)), J_2^H(a)\rangle +\end{aligned}$$

$$- \langle J_1(a), -S_{\gamma'(a)}(J_2^V(a))\rangle - \langle J_1(a), -A_{\gamma'(a)}(J_2^H(a))\rangle$$
$$= 0$$

where $J_1 \in \mathcal{J}_\gamma^{\text{hol}}$ and $J_2 \in \mathcal{J}_\gamma^{\text{proj}}$. Since the dimension of a subspace of Jacobi fields is determined by linearly independent choices for the initial conditions, clearly we have $\dim(\mathcal{J}_\gamma^{\text{proj}}) = \dim(T_{\gamma(a)}M) + \text{rank}(H) = 2n + k$.

It is worth to mention that the projectable Jacobi fields can be obtained by variations of γ through horizontal geodesics. The term "projectable" is due to the fact that each Jacobi field of $\mathcal{J}_\gamma^{\text{proj}}$ is π-related to a Jacobi field of $\mathcal{J}_{\pi\circ\gamma}$. In fact, there is a well defined projection map $\pi_* : \mathcal{J}_\gamma^{\text{proj}} \to \mathcal{J}_{\pi\circ\gamma}$ which is surjective and such that $\text{Ker}(\pi_*) = \mathcal{J}_\gamma^{\text{hol}}$.

In conclusion, a Finsler submersion offers to us an example where all the symplectic-types subspaces of Jacobi fields (isotropic $\mathcal{I}$, Lagrangian $\mathcal{L}$ and coisotropic $\mathcal{I}^\omega$) are present, as follows

$$\begin{array}{ccccc} \mathcal{I} & \subset & \mathcal{L} & \subset & \mathcal{I}^\omega \\ \| & & \| & & \| \\ \mathcal{J}_\gamma^{\text{hol}} & \subset & \mathcal{J}_\gamma^{M_b} & \subset & \mathcal{J}_\gamma^{\text{proj}} \end{array}$$

where M_b is the fiber of π through $b = \pi(\gamma(a))$. Furthermore, as consequence of the isomorphism theorem applied to π_*, we have the following nice geometric interpretation for the symplectic reduction

$$\frac{\mathcal{I}^\omega}{\mathcal{I}} = \frac{\mathcal{J}_\gamma^{\text{proj}}}{\mathcal{J}_\gamma^{\text{hol}}} \approx \mathcal{J}_{\pi\circ\gamma}.$$

5.3. Structures associated with isotropic subspaces of Jacobi fields

We stress in this section some technical results required to prove Lemma 5.4 in a manner that the reader can skip the proofs in this section without a further damage in the comprehension of this lemma and its proof.

More precisely we are going to generalize some aspects of the holonomy type Jacobi fields present in Example 5.3 to any isotropic subspace $\mathcal{I}$ of a given Jacobi triple (E, D, R). Two structures associated with a $\mathcal{I}$ will be explored, the $\mathcal{I}$-transverse Jacobi fields and the $\mathcal{I}$-Riccati operators. Although this generalization seems futile it will be quite useful in the proof of Lemma 5.4 where the choice of a specific isotropic subspace $\mathcal{I}$ is the key idea of the proof.

5.3.1. Horizontal bundle and transverse Jacobi fields

We start with a Lemma that describes the vertical bundle associated with an isotropic subspace $\mathcal{I}$.

Lemma 5.4. *Let $\mathcal{I} \subset \mathcal{J}$ be an isotropic subspace. Then*

(a) ***(Singular instants)*** *The set $\{t \in I \mid \mathcal{I}_t^0 \neq \{0\}\} \subset I$ is discrete.*
(b) ***(Vertical bundle)*** *The set*

$$V^{\mathcal{I}} := \coprod_{t\in I} \mathcal{I}(t) \oplus D(\mathcal{I}_t^0)(t)$$

is a vector subbundle of E with rank equal to the dimension of $\mathcal{I}$.

Proof. The proof is similar to the proof presented in Lemma 3.3 of [3]. □

Following the previous lemma, given an isotropic subspace $\mathcal{I}$, the set

$$I_{\mathcal{I}} := \{t \in I \mid \mathcal{I}_t^0 = \{0\}\} \tag{5.2}$$

will be called the set of *$\mathcal{I}$-regular instants* and in a logical contrast his complement will be called the set of *$\mathcal{I}$-singular instants* (item (a)).

Furthermore the item (b) of the lemma associates $\mathcal{I}$ to a vector subbundle $V^{\mathcal{I}} \subset E$ which will be called the *vertical subbundle* (associated with $\mathcal{I}$) and its orthogonal complement $H^{\mathcal{I}}$ will be called *horizontal subbundle* (associated with $\mathcal{I}$). The names of this subbundles are inspired by Example 5.3. When there is no risk of confusion, the spaces $V^{\mathcal{I}}$ and $H^{\mathcal{I}}$ will be simply denoted by V and H, respectively.

It is immediate from the item (b) that for each $t \in I_{\mathcal{I}}$ we have $\mathcal{I}(t) = V_t$. In particular for any Lagrangian subspace $\mathcal{L}$ we have $\mathcal{L}(t) = E_t$ for all $t \in I_{\mathcal{L}}$ by dimension issues.

Relatively to the decomposition $E = V \oplus H$ we denote the horizontal projection by π_H and the horizontal components of the operators D and R by D_H and $R_H := (R|_H)^H$ i.e., the H-component of the restriction of R to H. To deal with mixed components of the covariant derivative we define the tensor $\mathbb{A} : \Gamma(E) \to \Gamma(E)$ as

$$\mathbb{A}(X) := (DX^V)^H + (DX^H)^V;$$

cf. the definition of O'Neill tensor in Example 5.3.

It is straightforward to see that $\mathbb{A}$ is in fact a $C^\infty(I)$-homomorphism. Also $\mathbb{A}$ has the remarkable property that $\mathbb{A}_V = -\mathbb{A}_H^*$ which is proved in [3] and as a consequence

$$\mathbb{A}^2|_H = \mathbb{A}_V \mathbb{A}_H = -\mathbb{A}_H^* \mathbb{A}_H \tag{5.3}$$

is self-adjoint nonpositive operator.

Proposition 5.5 *(transverse Jacobi equation). Let $\mathcal{I} \subset \mathcal{J}$ be an isotropic subspace. Then*

- **(a)** *$(H, D_H, R_H - 3\mathbb{A}^2|_H)$ is a Jacobi triple.*
- **(b)** *$\operatorname{Ker}(\pi_{\mathrm{H}}|_{\mathcal{I}^\omega}) = \mathcal{I}$ and $\pi_{\mathrm{H}}(\mathcal{I}^\omega) = \mathcal{J}_{\mathcal{I}}$, where $\mathcal{J}_{\mathcal{I}}$ is the space of Jacobi fields associated to $(H, D_H, R_H - 3\mathbb{A}^2|_H)$.*
- **(c)** *$\pi_{\mathrm{H}} : \mathcal{I}^\omega/\mathcal{I} \longrightarrow \mathcal{J}_{\mathcal{I}}$ is a symplectic isomorphism.*

Proof. The item (a) is immediate, once $\mathbb{A}^2|_H$ is self-adjoint. Then we proceed with the proof of items (b) and (c).

(b) First we are going to prove that $\operatorname{Ker}(\pi_{\mathrm{H}}|_{\mathcal{I}^\omega}) = \mathcal{I}$. It is easy to check that $\operatorname{Ker}(\pi_{\mathrm{H}}|_{\mathcal{I}^\omega}) \supset \mathcal{I}$ so we are going to concentrate in prove that $\operatorname{Ker}(\pi_{\mathrm{H}}|_{\mathcal{I}^\omega}) \subset \mathcal{I}$. Given $[J] \in \operatorname{Ker}(\pi_{\mathrm{H}}|_{\mathcal{I}^\omega})$ the vector subspace $\widehat{\mathcal{I}} = \mathcal{I} + \mathbb{R}J$ is isotropic (since $J \in \mathcal{I}^\omega$) and for any $\widehat{\mathcal{I}}, \mathcal{I}$-regular $t \in I$

$$\dim(\mathcal{I} + \mathbb{R}J) \overset{(*)}{=} \dim(\widehat{\mathcal{I}}(t)) \overset{(**)}{=} \dim(\mathcal{I}(t)) \overset{(*)}{=} \dim(\mathcal{I}) \tag{5.4}$$

where the equality $(*)$ follows from the fact that t is regular and the equality $(**)$ follows from the fact that $J \in \operatorname{Ker}(\pi_H)$, i.e., $J(t)$ is vertical. Eq. (5.4) implies that $J \in \mathcal{I}$.

Now we are going to prove that $\pi_{\mathrm{H}}(\mathcal{I}^\omega) = \mathcal{J}_{\mathcal{I}}$. Note that it suffices to prove that $\pi_{\mathrm{H}}(\mathcal{I}^\omega) \subset \mathcal{J}_{\mathcal{I}}$ since

$$\operatorname{rank}(\pi_{\mathrm{H}}|_{\mathcal{I}^\omega}) = \dim(\mathcal{I}^\omega) - \dim(\operatorname{Ker}(\pi_{\mathrm{H}}|_{\mathcal{I}^\omega})) = 2\dim(H).$$

Also note that:
Claim Given $\widehat{J} \in \mathcal{I}^\omega$ there exists $J \in \mathcal{I}^\omega$ so that
(1) $J(t_0) = \widehat{J}^H(t_0)$, where t_0 is a $\mathcal{I}$-regular time.
(2) $[J] = [\widehat{J}]$, i.e., $\pi_H(J) = \pi_H(\widehat{J})$.
In fact, since t_0 is $\mathcal{I}$-regular, we have $\mathcal{I}(t_0) = V_{t_0}$ and there exists $\widetilde{J} \in \mathcal{I}$ such that $\widetilde{J}(t_0) = \widehat{J}^V(t_0)$. Set $J := \widehat{J} - \widetilde{J} \in \mathcal{I}^\omega$. Clearly $[J] = [\widehat{J}]$ and $J(t_0) = \widehat{J}(t_0) - \widetilde{J}(t_0) = \widehat{J}^H(t_0)$, and this concludes the proof of the claim.
Fix a $\mathcal{I}$-regular $t_0 \in I$, $u_{t_0} \in H_{t_0}$, $J \in \mathcal{I}^\omega$ such that $J(t_0) \in H_{t_0}$, $X \in \Gamma(H)$ D_H-parallel such that $X(t_0) = u_{t_0}$. Due to the above claim, in order to prove that $\pi_{\mathrm{H}}(\mathcal{I}^\omega) \subset \mathcal{J}_{\mathcal{I}}$ it suffices to prove Eq. (5.5) below.

$$\left\langle D_H^2 J^H(t_0), u_{t_0} \right\rangle = -\left\langle (R_H - 3\mathbb{A}^2|_H) J(t_0), u_{t_0} \right\rangle . \tag{5.5}$$

Let us accept for a moment the following two equations that we are going check later:

$$\langle J(t_0), D^2 X(t_0) \rangle = \langle J(t_0), \mathbb{A}^2(X)(t_0) \rangle. \tag{5.6}$$

$$\langle (DJ)^V(t_0), v \rangle = \langle \mathbb{A}^*(J^H(t_0)), v \rangle \, \forall v \in V_{t_0}. \tag{5.7}$$

Replacing Eq. (5.6) and Eq. (5.7) in the equation below (evaluated at $t = t_0$) we conclude the desired Eq. (5.5).

$$\begin{aligned}
\left\langle D_H^2 J^H, X \right\rangle &= \left\langle J^H, X \right\rangle'' \\
&= \langle J, X \rangle'' \\
&= \left\langle D^2 J, X \right\rangle + 2 \langle DJ, DX \rangle + \left\langle J, D^2 X \right\rangle \\
&= \langle -RJ, X \rangle + 2 \left\langle DJ, (DX)^V \right\rangle + \left\langle J, D^2 X \right\rangle \\
&= \left\langle (-RJ)^H, X \right\rangle + 2 \left\langle (DJ)^V, (\mathbb{A}_H X) \right\rangle + \left\langle J, D^2 X \right\rangle .
\end{aligned}$$

We now check Eq. (5.6).

$$\begin{aligned}
\langle J(t_0), D^2 X(t_0) \rangle &= \langle J(t_0), (D^2 X)^H(t_0) \rangle \\
&= \langle J(t_0), (D(DX)^V)^H(t_0) \rangle \\
&= \langle J(t_0), \mathbb{A}(DX)^V(t_0) \rangle \\
&= \langle J(t_0), \mathbb{A}\mathbb{A}X(t_0) \rangle
\end{aligned}$$

Finally we check Eq. (5.7). Consider $\widetilde{J} \in \mathcal{I}$ so that $\widetilde{J}(t_0) = v \in V_{t_0}$

$$\begin{aligned}
\langle (DJ)^V(t_0), \widetilde{J}(t_0) \rangle &= \langle DJ(t_0), \widetilde{J}(t_0) \rangle \\
&= \langle J(t_0), (D\widetilde{J})^H(t_0) \rangle \\
&= \langle J(t_0), \mathbb{A}\widetilde{J}(t_0) \rangle \\
&= \langle \mathbb{A}^* J(t_0), \widetilde{J}(t_0) \rangle \\
&= \langle \mathbb{A}^* J(t_0)^H, \widetilde{J}(t_0) \rangle.
\end{aligned}$$

(c) Fix a $\mathcal{I}$-regular $t_0 \in I$, $[J_1], [J_2] \in \mathcal{I}^\omega$ such that $J_1(t_0), J_2(t_0) \in H_{t_0}$ and $X_1, X_2 \in \Gamma(H)$ D_H-parallel such that $X_i(t_0) = J_i(t_0)$ for $i = 1, 2$. Then

$$\begin{aligned}\omega([J_1],[J_2]) &= \langle DJ_1(t_0), X_2(t_0)\rangle - \langle X_1(t_0), DJ_2(t_0)\rangle \\ &= \left(\left\langle J_1^H, X_2\right\rangle - \left\langle X_1, J_2^H\right\rangle\right)'(t_0) \\ &= \left\langle D_H J_1^H(t_0), X_2(t_0)\right\rangle - \left\langle X_1(t_0), D_H J_2^H(t_0)\right\rangle \\ &= \omega_{\mathcal{I}}(J_1^H, J_2^H),\end{aligned}$$

where $\omega_{\mathcal{I}}$ is the symplectic form associated to $(H, D_H, R_H - 3\mathbb{A}^2|_H)$. □

The space $\mathcal{J}_{\mathcal{I}}$ of Jacobi fields associated with the Jacobi triple $(H, D_H, R_H - 3\mathbb{A}^2|_H)$ will be called the *space of $\mathcal{I}$-transverse Jacobi fields.* As well as $\mathcal{J}$, the set of transverse Jacobi fields possess a symplectic form $\omega_{\mathcal{I}}$ (see Section 5.1). It is quite useful to mention that the Lagrangian subspaces of $(\mathcal{J}_{\mathcal{I}}, \omega_{\mathcal{I}})$ have a nice description which relates them to the Lagrangian subspaces of $\mathcal{J}$. This is the content of the subsequent corollary which follows as a consequence of the item (c) of the previous proposition.

Corollary 5.6 *(transverse Lagrangian subspaces). The map*

$$\{\mathcal{L} \in \Lambda(\mathcal{J}) \mid \mathcal{I} \subset \mathcal{L}\} \ni \mathcal{L} \longrightarrow \pi_H(\mathcal{L}/\mathcal{I}) \in \Lambda(\mathcal{J}_{\mathcal{I}})$$

is a bijection.

Remark 5.7. A geometric view of the transverse Jacobi vector fields, in a particular case, could be drawn from Example 5.3. Given a Finsler submersion $\pi : M \to B$ and a horizontal geodesic γ it was presented in that example a isomorphism between the symplectic reduction $\mathcal{J}_\gamma^{\text{proj}}/\mathcal{J}_\gamma^{\text{hol}}$ (quotient between projectable Jacobi fields and holonomy Jacobi fields) and the space $\mathcal{J}_{\pi\circ\gamma}$ of Jacobi fields along the projected geodesic. Then from item (c) of the previous proposition, the space of $\mathcal{J}_\gamma^{\text{hol}}$-transverse Jacobi fields is isomorphic to the $\mathcal{J}_{\pi\circ\gamma}$.

5.3.2. Riccati operators

Let $\mathcal{L} \subset \mathcal{J}$ be a Lagrangian subspace. Then for each $t \in I_{\mathcal{L}}$ the linear operator $S_t^{\mathcal{L}} : E_t \to E_t$ given by $S_t^{\mathcal{L}}(u_t) := DJ(t)$, where $J \in \mathcal{L}$ is such that $J(t) = u_t$, is well defined and self-adjoint, see [6]. Therefore it induces a $C^\infty(I_{\mathcal{L}})$-endomorphism in $\Gamma(E_{I_{\mathcal{L}}})$, the *Riccati operator* (associated with $\mathcal{L}$), which will be denoted by $S^{\mathcal{L}}$.

Lemma 5.8. *Assume that $I_{\mathcal{L}} = \mathbb{R}$. Let $\xi \in \mathbb{O}(E)$ be a D-parallel orthonormal frame. Then*

(a) *$[S^{\mathcal{L}}]_\xi$ is a solution of the Riccati differential equation (in the space of symmetric matrices $\mathbb{M}_n^{\text{sym}}(\mathbb{R})$)*

$$X' + X^2 + [R]_\xi = 0. \tag{5.8}$$

Moreover $\operatorname{Ker}(S^{\mathcal{L}} - D) = \mathcal{L}$.

(b) *Given $X : I_X \subset I \to \mathbb{M}_n^{\text{sym}}(\mathbb{R})$ a solution of Eq. (5.8) and denoting by S^X the $C^\infty(I_X)$-endomorphism in $\Gamma(E|_{I_X})$ characterized by $[S^X]_\xi = X$, the subspace $\mathcal{L} := \operatorname{Ker}(S^X - D) \subset \mathcal{J}$ is Lagrangian and $S^{\mathcal{L}} = S^X$.*

Proof. (a) It is immediate to see that $S^L = D$ in $\mathcal{L}$ from which follows that

$$-[R]_\xi[J]_\xi = [D^2 J]_\xi = [D(SJ)]_\xi = [S]_\xi'[J]_\xi + [S]_\xi[DJ]_\xi.$$

Then the fact that $\mathcal{L}(t) = E_t$ for all $t \in I_{\mathcal{L}}$ and the item (a) of Lemma 5.4 concludes the proof of this item.

(b) Since X is symmetric we have that S^X is self-adjoint so $\mathrm{Ker}(S^X - D)$ is isotropic. Furthermore $\dim(\mathrm{Ker}(S^X - D)) = \mathrm{rank}(E)$ (since $S^X - D$ is a differential operator of order 1). Then $\mathrm{Ker}(S^X - D)$ is in fact Lagrangian. □

Corollary 5.9. *Assume that $I_{\mathcal{L}} = \mathbb{R}$. The function $I_{\mathcal{L}} \ni t \to \mathrm{tr}(S^{\mathcal{L}}_t) \in \mathbb{R}$ is a solution for the following Riccati equation*

$$x' + x^2 + r = 0$$

where $r : I_{\mathcal{L}} \subset \mathbb{R} \to \mathbb{R}$ is given by $r = \mathrm{tr}(R) + (\mathrm{tr}(S^{\mathcal{L}^2}) - \mathrm{tr}(S^{\mathcal{L}})^2)$.

The previous lemma creates, for a fixed D-parallel orthonormal frame ξ, a bijection between the Lagrangian Grassmannian $\Lambda(\mathcal{J})$ (see Eq. (5.1)) and the space of solutions of the Riccati differential equation $X' + X^2 + [R]_\xi$ in the space of real symmetric matrices.

Various comparison result related to this type of Riccati differential equation was presented in [4]. Here we state a more weak result which will be useful for the proof of Wilking's decomposition lemma (Lemma 5.11).

Proposition 5.10. *Let $\mathcal{L} \subset \mathcal{J}$ be a Lagrangian subspace. If $I_{\mathcal{L}} = \mathbb{R}$ and $\mathrm{tr}(R) \geq 0$, then $\mathrm{tr}(R) = 0$ and $S^{\mathcal{L}}$ is identically 0.*

Proof. For the sake of completeness, let us briefly review the idea of the proof extracted from the proof of Theorem 1.7.1 of [6].

Define $r : I_{\mathcal{L}} \subset \mathbb{R} \to \mathbb{R}$ by $r = \mathrm{tr}(R) + (\mathrm{tr}(S^{\mathcal{L}^2}) - \mathrm{tr}(S^{\mathcal{L}})^2)$, as well as in Corollary 5.9. Since $\mathrm{tr}(R) \geq 0$ (by hypothesis) and in general $\mathrm{tr}(S^{\mathcal{L}^2}) \geq \mathrm{tr}(S^{\mathcal{L}})^2$, both summands in the definition of r are nonnegative and in particular $r \geq 0$. We state that in this case $\mathrm{tr}(S^{\mathcal{L}}) = 0$. Suppose by contradiction that $\mathrm{tr}(S^{\mathcal{L}}) \neq 0$ or more specifically there exists $t_0 \in I_{\mathcal{L}}$ such that $\mathrm{tr}(S^{\mathcal{L}}_{t_0}) \neq 0$. Without loss of generality, assume that $t_0 = 0$. Then $t \longmapsto \mathrm{tr}(S^{\mathcal{L}}_t)$ is a solution of the differential equation $x'(t) + x(t)^2 + r(t) = 0$ with a non null initial condition $x_0 = \mathrm{tr}(S^{\mathcal{L}}_{t_0})$ which implies that $\lim_{t \to -\frac{1}{x_0}^-} \mathrm{tr}(S^{\mathcal{L}}_t) = -\infty$, which is a contradiction. Finally Corollary 5.9 implies that $r = 0$ and by the definition of r we have that $\mathrm{tr}(R) = 0$ and $\mathrm{tr}(S^{\mathcal{L}^2}) = \mathrm{tr}(S^{\mathcal{L}})^2$ which occurs if only if $S^{\mathcal{L}} = \frac{1}{n}\mathrm{tr}(S^{\mathcal{L}})\mathrm{Id}$. Then $S^{\mathcal{L}} = 0$. □

Following what was presented in the previous section, given a isotropic subspace $\mathcal{I} \subset \mathcal{J}$, we can associate a Riccati operator $S^{\mathcal{L}_{\mathcal{I}}}$ to any $\mathcal{I}$-transverse Lagrangian subspace $\mathcal{L}_{\mathcal{I}}$ (i.e. a lagrangian subspace of $\mathcal{J}_{\mathcal{I}}$). Furthermore, by Corollary 5.6, $S^{\mathcal{L}_{\mathcal{I}}}$ can be associated with a Lagrangian subspace $\mathcal{L} \subset \mathcal{J}$, such that $\mathcal{L} \supset \mathcal{I}$.

The Riccati operator $S^{\mathcal{L}_{\mathcal{I}}}$ will be called a $\mathcal{I}$-transverse Riccati operator (associated to $\mathcal{L}_{\mathcal{I}}$) and the Lemma 5.8 and Proposition 5.10 holds to this type of Riccati operator either.

5.4. Wilking's decomposition lemma

We are finally ready to enunciate and prove the Wilking's decomposition lemma. It is worth to mention that, as noted at the beginning of the Section 5.3, the central idea of the proof of this lemma is the choice of the following specific isotropic subspace

$$\mathcal{I} = \mathrm{span}_{\mathbb{R}}\{J \in \mathcal{L} \mid J(t) = 0 \text{ for some } t \in \mathbb{R}\}$$

where $\mathcal{L}$ is a fixed Lagrangian subspace of a given Jacobi triple.

Lemma 5.11 *(Wilking's decomposition). Let (E, D, R) be a Jacobi triple, such that the base of E is $\mathbb{R}$ and R is nonnegative. Then*

$$\mathcal{L} = \operatorname{span}_{\mathbb{R}}\{J \in \mathcal{L} \mid J(t) = 0 \text{ for some } t \in \mathbb{R}\} \oplus \{J \in \mathcal{L} \mid J \text{ is parallel}\},$$

for all $\mathcal{L} \in \Lambda(\mathcal{J})$.

Proof. Define

$$\mathcal{I} := \operatorname{span}_{\mathbb{R}}\{J \in \mathcal{L} \mid J(t) = 0 \text{ for some } t \in \mathbb{R}\}. \tag{5.9}$$

It is immediate that $\mathcal{I} \subset \mathcal{J}$ is isotropic subspace of Jacobi fields. Then denote $k = \dim(\mathcal{I})$ and consider V and H the vertical and horizontal subbundles of E induced by $\mathcal{I}$. Also denote by π_H the horizontal projection whit respect to the decomposition $E = V \oplus H$ and by $\mathcal{L}_{\mathcal{I}} = \pi_H(\mathcal{L}) = \pi_H(\mathcal{L}/\mathcal{I})$ the $\mathcal{I}$-tranversal Lagrangian subspace induced by $\mathcal{I}$ which is explicitly described by this

$$\mathcal{L}_{\mathcal{I}} = \{J^H \in \mathcal{J}_{\mathcal{I}} \mid J \in \mathcal{L}\}$$

where $\mathcal{J}_{\mathcal{I}}$ is the space of $\mathcal{I}$-transverse Jacobi fields.

It is immediate that $\mathcal{I} \subset \mathcal{J}$ is isotropic subspace of Jacobi fields so that by Proposition 5.5 we have

$$\mathcal{L} \approx \mathcal{I} \oplus \mathcal{L}/\mathcal{I} \approx \mathcal{I} \oplus \mathcal{L}_{\mathcal{I}}.$$

Then it suffices to prove that $\mathcal{L}_{\mathcal{I}} = \{J \in \mathcal{L} \mid J \text{ is parallel}\}$. Note that $\mathcal{L}_{\mathcal{I}} \supset \{J \in \mathcal{L} \mid J \text{ is parallel}\}$. In fact for each $J \in \mathcal{L}$ parallel and $\tilde{J} \in \mathcal{I}$ we have that $\langle J, \tilde{J}\rangle' = \langle DJ, \tilde{J}\rangle + \langle J, D\tilde{J}\rangle = 2\langle DJ, \tilde{J}\rangle = 0$ and consequently $\langle J, \tilde{J}\rangle = 0$ i.e. J is horizontal.

In what follows we prove that $\mathcal{L}_{\mathcal{I}} \subset \{J \in \mathcal{L} \mid J \text{ is parallel}\}$ or equivalently that J^H is D-parallel Jacobi field in $\mathcal{L}$ for all $J \in \mathcal{L}$. First we need to prove the next two claims.

Claim A. $I_{\mathcal{L}_{\mathcal{I}}} = \mathbb{R}$ and the $\mathcal{I}$-transverse Riccati operator $S^{\mathcal{L}_{\mathcal{I}}}$ is identically 0.

Proof. First note that

$$\mathcal{I} = \{J \in \mathcal{L} \mid J(t) \in V_t \text{ for some } t \in \mathbb{R}\}. \tag{5.10}$$

In fact, if $J(t) \in V_t$ for some $t \in \mathbb{R}$, there exists $J_1 \in \mathcal{I}$ and $J_2 \in \mathcal{I}_t^0$ (i.e. $J_2(t) = 0$) such that $J(t) = J_1(t) + DJ_2(t)$ (see item (a) of Lemma 5.4). By multiplying both sides of the equation by $DJ_2(t)$ and using the fact that $\omega(J, J_2) = \omega(J_1, J_2) = 0$ we can infer that $\|DJ_2(t)\|^2 = 0$. We conclude that $J - J_1$ is a Jacobi field in $\mathcal{L}$ such that $(J - J_1)(t) = 0$ and by the definition of $\mathcal{I}$ (see Eq. (5.9)) $J - J_1 \in \mathcal{I}$ or equivalently $J \in \mathcal{I}$. The other inclusion follows from the fact that each $J \in \mathcal{I}$ is vertical.

Now take a $\mathcal{L}$-regular instant $t_0 \in \mathbb{R}$ and $J_1, \cdots, J_{n-k} \in \mathcal{L}$ such that $\{J_1^H(t_0), \cdots, J_{n-k}^H(t_0)\} \subset H_{t_0}$ is a base. Then $\{J_1^H, \cdots, J_{n-k}^H\}$ is a frame of H. In fact for each $t \in \mathbb{R}$ if $\sum \lambda_i J_i^H(t) = 0$ then $\sum \lambda_i J_i \in \mathcal{I}$ (by Eq. (5.10)) which implies that $\sum \lambda_i J_i^H(t_0) \in H_{t_0} \cap V_{t_0} = \{0\}$ and follows that $\lambda_i = 0$.

Finally, since H has a global frame of the form $\{J_1^H, \cdots, J_{n-k}^H\}$, we conclude that $\operatorname{span}_{\mathbb{R}}\{J_1^H, \cdots, J_{n-k}^H\} = \mathcal{L}_{\mathcal{I}}$ and consequently $\mathcal{L}_{\mathcal{I}}(t) = H_t$ which by definition of $L_{\mathcal{I}}$-regular instants means that $I_{\mathcal{L}_{\mathcal{I}}} = \mathbb{R}$ (see eq. (5.2) and remember that the total space for $\mathcal{I}$-transverse Jacobi fields is H).

Additionally, since R is nonnegative, Proposition 5.10 implies that $S^{\mathcal{L}_{\mathcal{I}}} \equiv 0$.

Claim B. $R_H, \mathbb{A} = 0$.

Proof. Since $S^{\mathcal{L}_{\mathcal{I}}} = 0$ (see **Claim A**), it follows by the transverse version of Riccati equation, i.e. $(S^{\mathcal{L}_{\mathcal{I}}})' + (S^{\mathcal{L}_{\mathcal{I}}})^2 + (R_H - 3\mathbb{A}^2|_H) = 0$ (see Eq (5.8)), that $R_H = 3\mathbb{A}^2|_H$. This equation together with Eq. (5.3) imply that $\forall X$

$$\begin{aligned}\langle R_H X, X\rangle &= \langle 3\mathbb{A}^2|_H X, X\rangle \\ &= -3\langle \mathbb{A}_H^* \mathbb{A}_H X, X\rangle \\ &= -3\langle \mathbb{A}_H X, \mathbb{A}_H X\rangle\end{aligned}$$

Therefore, since by hypothesis R_H is nonnegative, we infer that $R_H = 0$ and hence $\mathbb{A} = 0$.

Now we are going to prove that J^H is D-parallel for all $J \in \mathcal{L}$. Indeed J^H is D_H-parallel since $D_H J^H = S^{\mathcal{L}_\mathcal{I}} J^H = 0$ (see Claim A and item (a) of Proposition 5.8) which means that $(DJ^H)^H = 0$. The nullity of the vertical component of DJ^H follows from the fact that $\langle (DJ^H)^V, \tilde{J}\rangle = \langle \mathbb{A}J^H, \tilde{J}\rangle = 0$ for all $\tilde{J} \in \mathcal{I}$.

Now we proceed with the proof that J^H is a Jacobi field. Since J^H is D-parallel it suffices to prove that $RJ^H = 0$. By R self-adjointness and J^H D-parallelism, for each $\tilde{J} \in \mathcal{I}$, follows that

$$\begin{aligned}\left\langle (RJ^H)^V, \tilde{J}\right\rangle &= \left\langle RJ^H, \tilde{J}\right\rangle \\ &= \left\langle J^H, R\tilde{J}\right\rangle \\ &= \left\langle J^H, -D^2\tilde{J}\right\rangle \\ &= -\left\langle J^H, D\tilde{J}\right\rangle' \\ &= -\left\langle J^H, \tilde{J}\right\rangle'' \\ &= 0.\end{aligned}$$

Then by **Claim B** we conclude that $RJ^H = (RJ^H)^V + R_H J^H = 0$.

Finally we finish with the proof that $J^H \in \mathcal{L}$. Indeed it is immediately from J^H D-parallelism that $\omega(J^H, \tilde{J}) = -\langle J^H, \tilde{J}\rangle' = -\langle J^H, \tilde{J}^H\rangle' = 0$ for all $\tilde{J} \in \mathcal{L}$ which implies that $J^H \in \mathcal{L}^\omega = \mathcal{L}$. □

As a direct consequence of Wilking's decomposition lemma, we can infer Proposition 4.2.

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